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Euclidean Submanifolds via Tangential Components of Their Position Vector Fields

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Abstract: The position vector field is the most elementary and natural geometric object on a Euclidean submanifold. The position vector field plays important roles in physics, in particular in mechanics. For instance, in any equation of motion, the position vector $\mathbf{x}(t)$ is usually the most sought-after quantity because the position vector field defines the motion of a particle (i.e., a point mass): its location relative to a given coordinate system at some time variable t . This article is a survey article. The purpose of this article is to survey recent results of Euclidean submanifolds associated with the tangential components of their position vector fields. In the last section, we present some interactions between torqued vector fields and Ricci solitons.

Keywords: Euclidean submanifold; position vector field; concurrent vector field; concircular vector field; rectifying submanifold; T -submanifolds; constant ratio submanifolds; Ricci soliton

1. Introduction

For an n -dimensional submanifold M in the Euclidean m -space \mathbb{E}^m , the most elementary and natural geometric object is the position vector field \mathbf{x} of M . The position vector is a Euclidean vector $\mathbf{x} = \overrightarrow{OP}$ that represents the position of a point $P \in M$ in relation to an arbitrary reference origin $O \in \mathbb{E}^m$.

The position vector field plays important roles in physics, in particular in mechanics. For instance, in any equation of motion, the position vector $\mathbf{x}(t)$ is usually the most sought-after quantity because the position vector field defines the motion of a particle (i.e., a point mass): its location relative to a given coordinate system at some time variable t . The first and the second derivatives of the position vector field with respect to time t give the velocity and acceleration of the particle.

For a Euclidean submanifold M of a Euclidean m -space, there is a natural decomposition of the position vector field \mathbf{x} given by:

$$\mathbf{x} = \mathbf{x}^T + \mathbf{x}^N, \quad (1)$$

where \mathbf{x}^T and \mathbf{x}^N are the tangential and the normal components of \mathbf{x} , respectively. We denote by $|\mathbf{x}^T|$ and $|\mathbf{x}^N|$ the lengths of \mathbf{x}^T and of \mathbf{x}^N , respectively. Clearly, we have $|\mathbf{x}^N| = \sqrt{|\mathbf{x}|^2 - |\mathbf{x}^T|^2}$. In [1], the author provided a survey on several topics in differential geometry associated with position vector fields on Euclidean submanifolds.

In this paper, we discuss Euclidean submanifolds M whose tangential components \mathbf{x}^T admit some special properties such as concurrent, concircular, torse-forming, etc. Moreover, we will also discuss constant-ratio submanifolds, as well as Ricci solitons on Euclidean submanifolds with the potential fields of the Ricci solitons coming from the tangential components of the position vector fields. In the last section, we present some interactions between torqued vector fields and Ricci solitons.

2. Preliminaries

Let $x : M \rightarrow \mathbb{E}^m$ be an isometric immersion of a Riemannian manifold M into a Euclidean m -space \mathbb{E}^m . For each point $p \in M$, we denote by T_pM and $T_p^\perp M$ the tangent space and the normal space of M at p , respectively.

Let ∇ and $\tilde{\nabla}$ denote the Levi-Civita connections of M and \mathbb{E}^m , respectively. Then, the formulas of Gauss and Weingarten are given respectively by (cf. [2–6]):

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi, \tag{3}$$

for vector fields X, Y tangent to M and ξ normal to M , where h is the second fundamental form, D the normal connection and A the shape operator of M .

At a given point $p \in M$, the first normal space of M in \mathbb{E}^m , denoted by $\text{Im } h_p$, is the subspace given by:

$$\text{Im } h_p = \text{Span}\{h(X, Y) : X, Y \in T_pM\}. \tag{4}$$

For each normal vector ξ at p , the shape operator A_ξ is a self-adjoint endomorphism of T_pM . The second fundamental form h and the shape operator A are related by:

$$\langle A_\xi X, Y \rangle = \langle h(X, Y), \xi \rangle, \tag{5}$$

where $\langle \cdot, \cdot \rangle$ is the inner product on M , as well as on the ambient Euclidean space. The covariant derivative $\tilde{\nabla}h$ of h with respect to the connection on $TM \oplus T^\perp M$ is defined by:

$$(\tilde{\nabla}_X h)(Y, Z) = D_X(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{6}$$

For a given point $p \in M$, we put:

$$\text{Im } (\tilde{\nabla}h_p) = \{(\tilde{\nabla}_X h)(Y, Z) : X, Y, Z \in T_pM\}. \tag{7}$$

The subspace $\text{Im } \tilde{\nabla}h_p$ is called the second normal space at p .

The equation of Gauss of M in \mathbb{E}^m is given by:

$$R(X, Y; Z, W) = \langle h(X, W), h(Y, Z) \rangle - \langle h(X, Z), h(Y, W) \rangle \tag{8}$$

for X, Y, Z, W tangent to M , where R is the Riemann curvature tensors of M defined by:

$$R(X, Y; Z, W) = \langle \nabla_X \nabla_Y Z, W \rangle - \langle \nabla_Y \nabla_X Z, W \rangle - \langle \nabla_{[X, Y]} Z, W \rangle.$$

The equation of Codazzi is:

$$(\tilde{\nabla}_X h)(Y, Z) = (\tilde{\nabla}_Y h)(X, Z). \tag{9}$$

The mean curvature vector H of a submanifold M is defined by:

$$H = \frac{1}{n} \text{trace } h, \quad n = \dim M. \tag{10}$$

A Riemannian manifold is called a flat space if its curvature tensor R vanishes identically. Further, a submanifold M is called totally umbilical (respectively, totally geodesic) if its second fundamental form h satisfies $h(X, Y) = \langle X, Y \rangle H$ identically (respectively, $h = 0$ identically).

A hypersurface of a Euclidean $(n + 1)$ -space \mathbb{E}^{n+1} is called a quasi-umbilical hypersurface if its shape operator has an eigenvalue κ of multiplicity $mult(\kappa) \geq n - 1$ (cf. ([2], p. 147)). On the subset U of M on which $mult(\kappa) = n - 1$, an eigenvector with eigenvalue of multiplicity one is called a distinguished direction of the quasi-umbilical hypersurface.

The following lemmas can be found in [7].

Lemma 1. *Let $x : M \rightarrow \mathbb{E}^m$ be an isometric immersion of a Riemannian n -manifold into a Euclidean m -space \mathbb{E}^m . Then, $\mathbf{x} = \mathbf{x}^T$ holds identically if and only if M is a conic submanifold with the vertex at the origin.*

Lemma 2. *Let $x : M \rightarrow \mathbb{E}^m$ be an isometric immersion of a Riemannian n -manifold into \mathbb{E}^m . Then, $\mathbf{x} = \mathbf{x}^N$ holds identically if and only if M lies in a hypersphere centered at the origin.*

In view of Lemmas 1 and 2, we make the following.

Definition 1. *A submanifold M of \mathbb{E}^m is called proper if it satisfies $\mathbf{x} \neq \mathbf{x}^T$ and $\mathbf{x} \neq \mathbf{x}^N$ almost everywhere.*

3. Euclidean Submanifolds with Constant $|\mathbf{x}^T|$ or Constant $|\mathbf{x}^N|$

Euclidean submanifolds with constant $|\mathbf{x}^T|$ are called T -constant submanifolds in [8]. These submanifolds were first introduced and studied by the author in [8].

One important property of a T -constant proper hypersurface M is that the tangential component \mathbf{x}^T of the position vector field \mathbf{x} of M defines a principal direction for the hypersurface. Moreover, the normal component \mathbf{x}^N of M is nowhere zero (see, ([8], p. 66)).

T -constant Euclidean proper submanifolds were classified in [8] as follows.

Theorem 1. *Let $x : M \rightarrow \mathbb{E}^m$ be an isometric immersion of a Riemannian n -manifold into the Euclidean m -space. Then, M is a T -constant proper submanifold if and only if there exist real numbers a, b and local coordinate systems $\{s, u_2, \dots, u_n\}$ on M such that the immersion x is given by:*

$$x(s, u_2, \dots, u_n) = \sqrt{a^2 + b + 2as} Y(s, u_2, \dots, u_n), \tag{11}$$

where $Y = Y(s, u_2, \dots, u_n)$ satisfies the following conditions:

- (a) $Y = Y(s, u_2, \dots, u_n)$ lies in the unit hypersphere $S^{m-1}(1)$,
- (b) the coordinate vector field Y_s is perpendicular to coordinate vector fields Y_{u_2}, \dots, Y_{u_n} and
- (c) Y_s satisfies $|Y_s| = \sqrt{b + 2as} / (a^2 + b + 2as)$.

Now, we provide some examples of T -constant proper hypersurfaces in \mathbb{R}^{n+1} .

Example 1. *For a given real number $a > 0$ and for $s > 0$, we define $Y = Y(s, u_2, \dots, u_n)$ by:*

$$Y = \frac{1}{\sqrt{a^2 + 2as}} \left(a \sin \left(\frac{\sqrt{2as}}{a} \right) - \sqrt{2as} \cos \left(\frac{\sqrt{2as}}{a} \right), \right. \\ \left. \left\{ a \cos \left(\frac{\sqrt{2as}}{a} \right) + \sqrt{2as} \sin \left(\frac{\sqrt{2as}}{a} \right) \right\} \prod_{j=2}^n \cos u_j, \right. \\ \left. \left\{ a \cos \left(\frac{\sqrt{2as}}{a} \right) + \sqrt{2as} \sin \left(\frac{\sqrt{2as}}{a} \right) \right\} \sin u_2, \dots, \right. \\ \left. \left\{ a \cos \left(\frac{\sqrt{2as}}{a} \right) + \sqrt{2as} \sin \left(\frac{\sqrt{2as}}{a} \right) \right\} \sin u_n \prod_{j=2}^{n-1} \cos u_j \right) \tag{12}$$

in \mathbb{E}^{n+1} . Then, $|Y| = 1$, and $Y = Y(s, u_2, \dots, u_n)$ satisfies the conditions (a), (b) and (c) of Theorem 1. An easy computation shows that:

$$x = \sqrt{a^2 + 2as} Y(s, u_2, \dots, u_n) \tag{13}$$

satisfies $|x^T| = a$. Thus, (13) defines a proper T-constant submanifold in \mathbb{E}^{n+1} .

Similarly, one may also consider Euclidean submanifolds with constant $|x^N|$. Such submanifolds are called N-constant submanifolds in [8].

Proper N-constant Euclidean submanifolds were classified in [8] as follows.

Theorem 2. Let $x : M \rightarrow \mathbb{E}^m$ be an isometric immersion of a Riemannian n-manifold into the Euclidean m-space. Then, M is an N-constant proper submanifold if and only if there exist a positive number c and local coordinate systems $\{s, u_2, \dots, u_n\}$ on M such that the immersion x is given by:

$$x(s, u_2, \dots, u_n) = \sqrt{s^2 + c^2} Y(s, u_2, \dots, u_n), \tag{14}$$

where $Y = Y(s, u_2, \dots, u_n)$ satisfies the conditions:

- (1) $Y = Y(s, u_2, \dots, u_n)$ lies in the unit hypersphere $S^{m-1}(1)$,
- (2) Y_s is perpendicular to coordinate vector fields Y_{u_2}, \dots, Y_{u_n} and
- (3) Y_s satisfies $|Y_s| = c / (s^2 + c^2)$.

Here are some examples of N-constant proper hypersurfaces of \mathbb{E}^{n+1} .

Example 2. For a given positive numbers c, we define:

$$Y = \frac{1}{\sqrt{s^2 + c^2}} \left(c, s \prod_{j=2}^n \cos u_j, s \sin u_2, \dots, s \sin u_n \prod_{j=2}^{n-1} \cos u_j \right) \tag{15}$$

in \mathbb{E}^{n+1} . Then, $\langle Y, Y \rangle = 1$, and $Y = Y(s, u_2, \dots, u_n)$ satisfies the conditions (1), (2) and (3) of Theorem 2.

An easy computation shows that:

$$x = \sqrt{s^2 + c^2} Y(s, u_2, \dots, u_n) \tag{16}$$

satisfies $\langle x^N, x^N \rangle = c^2$, which provides an example of a proper N-constant submanifold.

4. Euclidean Submanifolds with Constant Ratio $|x^T| : |x^N|$

Euclidean submanifolds with the ratio $|x^T| : |x^N|$ being constant are called constant ratio submanifolds. The study of such submanifolds was initiated by the author in [9,10].

As we mentioned in [1], constant-ratio curves in a plane are exactly the equiangular curves in the sense of D’Arcy Thompson’s biology theory on growth and form [11]. Thus, constant-ratio submanifolds can be regarded as a higher dimensional version of Thompson’s equiangular curves. For this reason, constant-ratio submanifolds are also known in some literature as equiangular submanifolds (see, e.g., [12,13]).

Constant-ratio submanifolds were completely classified by the author in [9,10] as follows.

Theorem 3. Let $x : M \rightarrow \mathbb{E}^m$ be an isometric immersion of a Riemannian n-manifold into the Euclidean m-space. Then, M is a constant-ratio proper submanifold if and only if there exists a number $b \in (0, 1)$ and local coordinate systems $\{s, u_2, \dots, u_n\}$ on M such that the immersion x is given by:

$$x(s, u_2, \dots, u_n) = bs Y(s, u_2, \dots, u_n), \tag{17}$$

where $Y = Y(s, u_2, \dots, u_n)$ satisfies the conditions:

- (a) $Y = Y(s, u_2, \dots, u_n)$ lies in the unit hypersphere $S^{m-1}(1)$,
- (b) Y_s is perpendicular to Y_{u_2}, \dots, Y_{u_n} and
- (c) $|Y_s| = \sqrt{1 - b^2} / (bs)$.

We give the following examples of constant-ratio hypersurfaces.

Example 3. Let b be a real number in $(0, 1)$ and $s > 0$. We define:

$$Y(s, u_2, \dots, u_n) = \left(\sin \left(\frac{\sqrt{1 - b^2}}{b} \ln s \right), \cos \left(\frac{\sqrt{1 - b^2}}{b} \ln s \right) \prod_{j=2}^n \cos u_j, \right. \\ \left. \cos \left(\frac{\sqrt{1 - b^2}}{b} \ln s \right) \sin u_2, \dots, \cos \left(\frac{\sqrt{1 - b^2}}{b} \ln s \right) \sin u_n \prod_{j=2}^{n-1} \cos u_j \right)$$

in \mathbb{E}^{n+1} . Then, $|Y| = 1$, and $Y = Y(s, u_2, \dots, u_n)$ is a local parametrization of the unit sphere S^n . Moreover, $Y(s, u_2, \dots, u_n)$ satisfies Conditions (b) and (c) of Theorem 3.

An easy computation shows that $x(s, u_2, \dots, u_n) = bsY(s, u_2, \dots, u_n)$ satisfies $|x| = bs$ and $|x^T| = b^2s$. Hence, $|x^T| = b|x|$. Consequently, x defines a constant-ratio hypersurface in \mathbb{E}^{n+1} .

Remark 1. Constant-ratio curves also relate to the motion in a central force field that obeys the inverse-cube law. In fact, the trajectory of a mass particle subject to a central force of attraction located at the origin that obeys the inverse-cube law is a curve of constant-ratio. The inverse-cube law was originated from Sir Isaac Newton (1642–1727) in his letter sent on 13 December 1679 to Robert Hooke (1635–1703). This letter is of great historical importance since it reveals the state of Newton’s development of dynamics at that time (see, for instance, [14,15], pp. 266–271, [16,17], Book I, Section II, Proposition IX).

Let ρ denote the distance function of a submanifold M in \mathbb{E}^m , i.e., $\rho = |x|$. It was proven in [18] that the Euclidean submanifold M is of constant-ratio if and only if the gradient of the distance function ρ has constant length.

Remark 2. Constant ratio submanifolds are related to the notion of convolution manifolds introduced by the author in [18,19], as well.

5. Rectifying Euclidean Submanifolds with Concurrent x^T

Let $\alpha : I \rightarrow \mathbb{E}^3$ be a unit speed curve in the Euclidean three-space \mathbb{E}^3 with Frenet–Serret apparatus $\{\kappa, \tau, T, N, B\}$, where κ , τ , T , N and B denote the curvature, the torsion, the unit tangent T , the unit principal normal N and the unit binormal of α , respectively. Then, α is called a Frenet curve if the curvature and torsion of α satisfy $\kappa > 0$ and $\tau \neq 0$.

The famous Frenet formulas of α are given by:

$$\begin{cases} \mathbf{t}' = \kappa \mathbf{n}, \\ \mathbf{n}' = -\kappa \mathbf{t} + \tau \mathbf{b}, \\ \mathbf{b}' = -\tau \mathbf{n}. \end{cases} \tag{18}$$

At each point of the curve, the planes spanned by $\{\mathbf{t}, \mathbf{n}\}$, $\{\mathbf{t}, \mathbf{b}\}$ and $\{\mathbf{n}, \mathbf{b}\}$ are known as the osculating plane, the rectifying plane and the normal plane, respectively.

It is well known in elementary differential geometry that a curve in \mathbb{E}^3 lies in a plane if its position vector x lies in its osculating plane at each point; and it lies on a sphere if its position vector lies in its normal plane at each point. In view of these basic facts, the author defined a rectifying curve in \mathbb{E}^3 as a Frenet curve whose position vector field always lie in its rectifying plane [20]. Moreover, he completely classified in [20] rectifying curves in \mathbb{E}^3 . Furthermore, he proved in [21] that a curve on a general

cone (not necessarily a circular one) in \mathbb{E}^3 is a geodesic if and only if it is a rectifying curve or an open portion of a ruling of the cone. In [22], several interesting links between rectifying curves, centrodes and extremal curves were established by B.-Y. Chen and F. Dillen. Some further results in this respect were also obtained recently in [23,24].

Clearly, it follows from the definition of a rectifying curve $\alpha : I \rightarrow \mathbb{E}^3$ that the position vector field \mathbf{x} of α satisfies:

$$\mathbf{x}(s) = \lambda(s)\mathbf{t}(s) + \mu(s)\mathbf{b}(s) \tag{19}$$

for some functions λ and μ .

For a Frenet curve $\gamma : I \rightarrow \mathbb{E}^3$, the first normal space of γ at s_0 is the line spanned by the principal normal vector $\mathbf{n}(s_0)$. Hence, the rectifying plane of γ at s_0 is nothing but the plane orthogonal to the first normal space at s_0 . For this reason, for a submanifold M of \mathbb{E}^m and a point $p \in M$, we call the subspace of $T_p\mathbb{E}^m$ the rectifying space of M at p if it is the orthogonal complement to the first normal space $\text{Im } \sigma_p$.

According to [7], a submanifold M of a Euclidean m -space \mathbb{E}^m is called a rectifying submanifold if the position vector field \mathbf{x} of M always lies in its rectifying space. In other words, M is called a rectifying submanifold if and only if:

$$\langle \mathbf{x}(p), \text{Im } h_p \rangle = 0 \tag{20}$$

holds for each point $p \in M$. A non-trivial vector field Z on a Riemannian manifold M is called concurrent if it satisfies $\nabla_X Z = X$ for any vector X tangent to M , where ∇ is the Levi-Civita connection of M (cf. [25–28]).

The following results on rectifying submanifolds were proven in [7,29].

Theorem 4. *If M is a proper submanifold of \mathbb{E}^m , then M is a rectifying submanifold if and only if \mathbf{x}^T is a concurrent vector field on M .*

Theorem 5. *A proper hypersurface M of \mathbb{E}^{n+1} is rectifying if and only if M is an open portion of a hyperplane L of \mathbb{E}^{n+1} with $o \notin L$, where zero denotes the origin of \mathbb{E}^{n+1} .*

Theorem 6. *Let M be a rectifying proper submanifold of \mathbb{E}^m . If $m \geq 2 + \dim M$, then with respect to some suitable local coordinate systems $\{s, u_2, \dots, u_n\}$ on M , the immersion x of M in \mathbb{E}^m takes the form:*

$$x(s, u_2, \dots, u_n) = \sqrt{s^2 + c^2} Y(s, u_2, \dots, u_n), \quad \langle Y, Y \rangle = 1, \quad c > 0, \tag{21}$$

such that the metric tensor g_Y of the spherical submanifold defined by Y satisfies:

$$g_Y = \frac{c^2}{(s^2 + c^2)^2} ds^2 + \frac{s^2}{s^2 + c^2} \sum_{i,j=2}^n g_{ij}(u_2, \dots, u_n) du_i du_j. \tag{22}$$

Conversely, the immersion defined by (21) and (22) is a rectifying proper submanifold.

Remark 3. *For the pseudo-Euclidean version of Theorem 5, see [30].*

6. Euclidean Submanifolds with Conircular \mathbf{x}^T

A non-trivial vector field Z on a Riemannian manifold M is called a conircular vector field if it satisfies (cf., e.g., [5,31,32]):

$$\nabla_X Z = \varphi X, \quad X \in TM, \tag{23}$$

where φ is a smooth function on M , called the concircular function. Obviously, a concircular vector field with $\varphi = 1$ is a concurrent vector field. For simplicity, we call a Euclidean submanifold with concircular \mathbf{x}^T a circular submanifold.

The following result from [33] classifies concircular submanifolds completely.

Theorem 7. *Let M be a proper submanifold of a Euclidean m -space \mathbb{E}^m with origin zero. If $n = \dim M \geq 2$, then M is a concircular submanifold if and only if one of the following three cases occurs:*

- (a) M is an open portion of a linear n -subspace L^n of \mathbb{E}^m such that $o \notin L$.
- (b) M is an open portion of a hypersphere S^n of a linear $(n + 1)$ -subspace L^{n+1} of \mathbb{E}^m such that the origin of \mathbb{E}^m is not the center of S^n .
- (c) $m \geq n + 2$. Moreover, with respect to some suitable local coordinate systems $\{s, u_2, \dots, u_n\}$ on M , the immersion x of M in \mathbb{E}^m takes the following form:

$$x(s, u_2, \dots, u_n) = \sqrt{2\rho} Y(s, u_2, \dots, u_n), \quad \langle Y, Y \rangle = 1, \tag{24}$$

where $Y : M \rightarrow S_0^{m-1}(1) \subset \mathbb{E}^m$ is an immersion of M into the unit hypersphere $S_0^{m-1}(1)$ such that the induced metric g_Y via Y is given by:

$$g_Y = \frac{2\rho - \rho'^2}{4\rho^2} ds^2 + \frac{\rho'^2}{2\rho} \sum_{i,j=2}^n g_{ij}(u_2, \dots, u_n) du_i du_j. \tag{25}$$

where $\rho = \rho(s)$ satisfies $2\rho > \rho'^2 > 0$ on an open interval I .

Next, we provide one explicit example of a concircular surface in \mathbb{E}^4 .

Example 4. *If we choose $n = 2$ and $\rho(s) = \frac{3}{8}s^2$, then (33) reduces to:*

$$g_Y = \frac{1}{3s^2} ds^2 + \frac{3}{4} du^2. \tag{26}$$

Let us define $Y : I_1 \times I_2 \rightarrow S_0^3(1) \subset \mathbb{E}^4$ to be the map of $I_1 \times I_2$ into $S_0^3(1)$ given by:

$$Y(s, u) = \frac{1}{\sqrt{2}} \left(\cos \left(\frac{\sqrt{2}}{\sqrt{3}} \ln s \right), \sin \left(\frac{\sqrt{2}}{\sqrt{3}} \ln s \right), \cos \left(\frac{\sqrt{6}}{2} u \right), \sin \left(\frac{\sqrt{6}}{2} u \right) \right). \tag{27}$$

Then, the induced metric tensor of $I_1 \times I_2$ via the map Y is given by (34). Therefore, $P^2 = (I_1 \times I_2, g_Y)$ with the induced metric tensor g_Y being a flat surface.

Now, consider $x(s, u) : I_1 \times I_2 \rightarrow \mathbb{E}^4$ given by $x(s, u) = F(s)Y(s, u)$, i.e.,

$$x(s, u) = \frac{\sqrt{3}s}{2\sqrt{2}} \left(\cos \left(\frac{\sqrt{2}}{\sqrt{3}} \ln s \right), \sin \left(\frac{\sqrt{2}}{\sqrt{3}} \ln s \right), \cos \left(\frac{\sqrt{6}}{2} u \right), \sin \left(\frac{\sqrt{6}}{2} u \right) \right). \tag{28}$$

Then, it is easy to verify that the induced metric via x is:

$$g = ds^2 + \frac{9}{16} s^2 du^2. \tag{29}$$

Hence, the Levi-Civita connection of $M = (I_1 \times I_2, g)$ satisfies:

$$\nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial s} = 0, \quad \nabla_{\frac{\partial}{\partial u}} \frac{\partial}{\partial s} = \frac{1}{s} \frac{\partial}{\partial u}. \tag{30}$$

Using (28) and (29), it is easy to verify that the tangential component $\mathbf{x}^T = \frac{3}{4}s\frac{\partial}{\partial s}$ of the position vector field \mathbf{x} is a concircular vector field satisfying $\nabla_Z\mathbf{x}^T = \frac{3}{4}Z$ for $Z \in TM$. Consequently, M is a concircular surface in \mathbb{E}^4 .

Remark 4. *Concircular vector fields play some important roles in general relativity. For instance, it was proven in [34] that a Lorentzian manifold is a generalized Robertson–Walker spacetime if and only if it admits a timelike concircular vector field. For the most recent surveys on generalized Robertson–Walker spacetimes, see [5,35].*

Remark 5. *It was proven in [36] that every Kaehler manifold M (or more generally, pseudo-Kaehler manifold) with $\dim_{\mathbb{C}} M > 1$ does not admit a non-trivial concircular vector field.*

7. Euclidean Submanifolds with Torse-Forming \mathbf{x}^T

In [37], K. Yano extended concurrent and concircular vector fields to torse-forming vector fields. According to K. Yano, a vector field v on a Riemannian (or pseudo-Riemannian) manifold M is called a torse-forming vector field if it satisfies:

$$\nabla_X v = \varphi X + \alpha(X)v, \quad \forall X \in TM, \tag{31}$$

for a function φ and a one-form α on M . The one-form α is called the generating form, and the function φ is called the conformal scalar (see [38]). A torqued vector field is a torse-forming vector field v satisfying (31) with $\alpha(v) = 0$ (see [39,40]).

Generalized Robertson–Walker (GRW) spacetimes were introduced by L. J. Alías, A. Romero and M. Sánchez in [41]. The first author proved in [34] that a Lorentzian manifold is a GRW spacetime if and only if it admits a time-like concircular vector field. For further results in this respect, see an excellent survey on GRW spacetimes by C. A. Mantica and L. G. Molinari [35] (see also [5]).

Twisted products are natural extensions of warped products in which the warping functions were replaced by twisting functions (cf. [5,42]). It was proven in [39] that a Lorentzian manifold is a twisted space of the form $I \times_{\lambda} F$ with time-like base I if and only if it admits a time-like torqued vector field. Recently, C. A. Mantica and L. G. Molinari proved in [43] that such a Lorentzian twisted space can also be characterized as a Lorentzian manifold admitting a torse-forming time-like unit vector field.

Before we state the results for Euclidean hypersurfaces with torse-forming \mathbf{x}^T , we give the following simple link between Hessian of functions and torse-forming vector fields.

Theorem 8. *Let f be a non-constant function on a Riemannian manifold M . Then, the gradient ∇f of f is a torse-forming vector field if and only if the Hessian H^f satisfies:*

$$H^f = \varphi g + \gamma df \otimes df \tag{32}$$

where φ and γ are functions on M .

Proof. Let f be a non-constant function on a Riemannian manifold M . Assume that the gradient ∇f of f is a torse-forming vector field so that:

$$\nabla_X(\nabla f) = \varphi X + \alpha(X)\nabla f \tag{33}$$

for some function φ and one-form α on M . Then, for any vector fields X, Y on M , the Hessian H^f of f satisfies:

$$\begin{aligned} H^f(X, Y) &= XYf - (\nabla_X Y)f = X \langle Y, \nabla f \rangle - \langle \nabla_X Y, \nabla f \rangle \\ &= \langle Y, \nabla_X(\nabla f) \rangle = \varphi \langle Y, X \rangle + \alpha(X) \langle Y, \nabla f \rangle \\ &= \varphi \langle Y, X \rangle + \alpha(X)df(Y). \end{aligned} \tag{34}$$

Since the Hessian $H^f(X, Y)$ is symmetric in X and Y , we derive from (34) that:

$$\begin{aligned} \langle Y, \nabla_X(\nabla f) \rangle &= X \langle Y, \nabla f \rangle - \langle \nabla_X Y, \nabla f \rangle = H^f(X, Y) \\ &= \langle Y, \varphi, X + \gamma df(X) \nabla f \rangle \end{aligned}$$

for vector field X, Y . Therefore, we obtain (32) with $\alpha = \gamma df$. Consequently, the gradient ∇f of f is a torse-forming vector field. \square

The following corollary is an easy consequence of Theorem 8.

Corollary 1. *Let f be a non-constant function on a Riemannian manifold M . If the gradient ∇f of f is a torqued vector field, then it is a concircular vector field on M .*

Remark 6. *Theorem 8 extends Lemma 4.1 of [31].*

Next, we present the following results from [44] for Euclidean hypersurfaces with torse-forming \mathbf{x}^T .

Proposition 1. *Let M be a proper hypersurface of \mathbb{E}^m . If the tangential component \mathbf{x}^T of the position vector field \mathbf{x} of M is a torse-forming vector field, then M is a quasi-umbilical hypersurface with \mathbf{x}^T as its distinguished direction.*

For quasi-umbilical hypersurfaces in \mathbb{E}^m we refer to [2,45].

A rotational hypersurface $M = \gamma \times S^{n-1}$ in \mathbb{E}^{n+1} is an $O(n - 1)$ -invariant hyper-surface, where S^{n-1} is a Euclidean sphere and:

$$\gamma(x) = (x, g(x)), \quad g(x) > 0, \quad x \in I, \tag{35}$$

is a plane curve (the profile curve) defined on an open interval I and the x -axis is called the axis of rotation. The rotational hypersurface M can be expressed as:

$$\mathbf{x} = (u, g(u)y_1, \dots, g(u)y_n) \quad \text{with} \quad y_1^2 + \dots + y_n^2 = 1. \tag{36}$$

The hypersurfaces are called a spherical cylinder if its profile curve γ is a horizontal line segment (i.e., $g = \text{constant} \neq 0$). Additionally, it is called a spherical cone if γ is a non-horizontal line segment (i.e., $g = cu, 0 \neq c \in \mathbf{R}$). For simplicity, we only consider rotational hypersurfaces M , which contain no open parts of hyperspheres, spherical cylinders or spherical cones.

A torse-forming vector field v is called proper torse-forming if the one-form α in (31) is nowhere zero on a dense open subset of M .

The simple link between rotational hypersurfaces and torse-forming \mathbf{x}^T is the following.

Theorem 9. *Let M be a proper hypersurface of \mathbb{E}^{n+1} with $n \geq 3$. Then, the tangential component \mathbf{x}^T of the position vector field \mathbf{x} of M is a proper torse-forming vector field if and only if M is an open part of a rotational hypersurface whose axis of rotation contains the origin [44].*

8. Rectifying Submanifolds of Riemannian Manifolds

In [39], the notion of rectifying submanifolds of Euclidean spaces was extended to rectifying submanifolds of Riemannian manifolds.

Definition 2. *Let V be a non-vanishing vector field on a Riemannian manifold \tilde{M} , and let M be a submanifold of \tilde{M} such that the normal component V^N of V is nowhere zero on M . Then, M is called a rectifying submanifold (with respect to V) if and only if:*

$$\langle V(p), \text{Im } h_p \rangle = 0 \tag{37}$$

holds at each $p \in M$.

Definition 3. A submanifold M of a Riemannian manifold \tilde{M} is said to be twisted if:

$$\text{Im } \bar{\nabla} h_p \not\subseteq \text{Im } h_p \tag{38}$$

holds at each point $p \in M$.

A vector field on a Riemannian manifold M is called a gradient vector field if it is the gradient ∇f of some function f on M .

In terms of gradient vector fields, Corollary 1 can be restated as the follows.

Proposition 2. If a torqued vector field on a Riemannian manifold M is a gradient vector field, then it is a concircular vector field.

The following result from [39] is an extension of Theorem 4.

Theorem 10. Let M be a submanifold of a Riemannian manifold \tilde{M} , which admits a torqued vector field \mathcal{T} . If the tangential component \mathcal{T}^T of \mathcal{T} is nonzero on M , then M is a rectifying submanifold (with respect to \mathcal{T}) if and only if \mathcal{T}^T is torse-forming vector field on M whose conformal scalar is the restriction of the torqued function and whose generating form is the restriction of the torqued form of \mathcal{T} on M .

In [39], we also have the following results.

Theorem 11. Let M be a submanifold of a Riemannian manifold \tilde{M} endowed with a concircular vector field $Z \neq 0$ with $Z^T \neq 0$ on M . Then, M is a rectifying submanifold with respect to Z if and only if the tangential component Z^T of Z is a concircular vector field with the concircular function given by the restriction of the concircular function of Z on M .

The following result is an immediate consequence of Theorem 11.

Corollary 2. Let M be a submanifold of a Riemannian manifold \tilde{M} endowed with a concurrent vector field $Z \neq 0$ such that $Z^T \neq 0$ on M . Then, M is a rectifying submanifold with respect to Z if and only if the tangential component Z^T of Z is a concurrent vector field on M .

Moreover, from Theorem 11, we also have the following.

Proposition 3. Let \tilde{M} be a Riemannian m -manifold endowed with a concircular vector field Z . If M is a rectifying submanifold of \tilde{M} with respect to Z , then we have:

- (1) Z^N is of constant length $\neq 0$.
- (2) The concircular function φ of Z^T is given by $\varphi = Z^T(\ln \rho)$, where $\rho = |Z^T|$.

9. Euclidean Submanifolds with x^T as Potential Fields

A smooth vector field ζ on a Riemannian manifold (M, g) is said to define a Ricci soliton if it satisfies:

$$\frac{1}{2} \mathcal{L}_\zeta g + Ric = \lambda g, \tag{39}$$

where $\mathcal{L}_\zeta g$ is the Lie-derivative of the metric tensor g with respect to ζ , Ric is the Ricci tensor of (M, g) and λ is a constant (cf. for instance [46–48]). We shall denote a Ricci soliton by (M, g, ζ, λ) .

A Ricci soliton (M, g, ζ, λ) is called shrinking, steady or expanding according to $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, respectively. A trivial Ricci soliton is one for which ζ is zero or Killing, in which case the metric is Einstein.

A Ricci soliton (M, g, ζ, λ) is called a gradient Ricci soliton if its potential field ζ is the gradient of some smooth function f on M .

For a gradient Ricci soliton, the soliton equation can be expressed as:

$$Ric_f = \lambda g, \tag{40}$$

where

$$Ric_f := Ric + Hess(f) \tag{41}$$

is known as the Bakry–Émery curvature, where $Hess(f)$ denotes the Hessian of f . Hence, a gradient Ricci soliton has constant Bakry–Émery curvature; a similar role as an Einstein manifold.

Compact Ricci solitons are the fixed points of the Ricci flow:

$$\frac{\partial g(t)}{\partial t} = -2Ric(g(t)) \tag{42}$$

projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings and often arise as blow-up limits for the Ricci flow on compact manifolds. Further, Ricci solitons model the formation of singularities in the Ricci flow, and they correspond to self-similar solutions (cf. [47]).

During the last two decades, the geometry of Ricci solitons has been the focus of attention of many mathematicians. In particular, it has become more important after Grigory Perelman [48] applied Ricci solitons to solve the long-standing Poincaré conjecture posed in 1904. G. Perelman observed in [48] that the Ricci solitons on compact simply connected Riemannian manifolds are gradient Ricci solitons as solutions of Ricci flow.

The next result from ([31], Theorem 5.1) classifies Ricci solitons with concircular potential field.

Theorem 12. *A Ricci soliton (M, g, v, λ) on a Riemannian n -manifold (M, g) with $n \geq 3$ has concircular potential field v if and only if the following three conditions hold:*

- (a) *The function φ in (31) is a nonzero constant, say b ;*
- (b) *$\lambda = b$;*
- (c) *M is an open portion of a warped product manifold $I \times_{bs+c} F$, where I is an open interval with arc-length s , c is a constant and F is an Einstein $(n - 1)$ -manifold whose Ricci tensor satisfies*

$$Ric_F = (n - 2)b^2 g_F,$$

where g_F is the metric tensor of F .

By combining Theorem 12 with some results from [31], we have the following .

Corollary 3. *The only Riemannian manifold of constant sectional curvature admitting a Ricci soliton with concircular potential field is a Euclidean space [31].*

Now, we present results on Ricci solitons of Euclidean hypersurfaces such that the potential field ζ is the tangential components \mathbf{x}^T of the position vector field of the hypersurfaces.

For Ricci solitons on a Euclidean submanifold with the potential field given by \mathbf{x}^T , we have the following result from ([49], Theorem 4.1, p. 6).

Theorem 13. *Let (M, g, ζ, λ) be a Ricci soliton on a Euclidean submanifold M of \mathbb{E}^m . If the potential field ζ is the tangential component \mathbf{x}^T of the position vector field of M , then the Ricci tensor of (M, g) satisfies:*

$$Ric(X, Y) = (\lambda - 1) \langle X, Y \rangle - \langle h(X, Y), \mathbf{x}^N \rangle \tag{43}$$

for any X, Y tangent to M .

Let ζ be a normal vector field of a Riemannian submanifold M . Then, M is called ζ -umbilical if its shape operator satisfies $A_\zeta = \varphi I$, where φ is a function on M and I is the identity map.

The following are some simple applications of Theorem 13.

Corollary 4. A Ricci soliton $(M, g, \mathbf{x}^T, \lambda)$ on a Euclidean submanifold M is trivial if and only if M is \mathbf{x}^\perp -umbilical.

Corollary 5. Every Ricci soliton $(M, g, \mathbf{x}^T, \lambda)$ on a totally umbilical submanifold M of \mathbb{E}^m is a trivial Ricci soliton.

Corollary 6. If $(M, g, \mathbf{x}^T, \lambda)$ is a Ricci soliton on a minimal submanifold M in \mathbb{E}^m , then M has constant scalar curvature given by $\frac{1}{2}n(\lambda - 1)$ with $n = \dim M$.

Corollary 7. Every Ricci soliton $(M, g, \mathbf{x}^T, \lambda)$ on a Euclidean submanifold M is a gradient Ricci soliton with potential function $\varphi = \frac{1}{2}\tilde{g}(\mathbf{x}, \mathbf{x})$.

The next result was also obtained in ([49], Proposition 4.1, p. 6).

Theorem 14. If (M, g, ζ, λ) is a Ricci soliton on a hypersurface of M of \mathbb{E}^{n+1} whose potential field ζ is \mathbf{x}^T , then M has at most two distinct principal curvatures given by:

$$\kappa_1, \kappa_2 = \frac{n\alpha + \rho \pm \sqrt{(n\alpha + \rho)^2 + 4 - 4\lambda}}{2}, \tag{44}$$

where α is the mean curvature and ρ is the support function of M , i.e., $\rho = \langle \mathbf{x}, N \rangle$ and $H = \alpha N$ with N being a unit normal vector field.

The following result from ([50], Theorem 4.2) classifies the Ricci soliton of Euclidean hypersurfaces with the potential field given by \mathbf{x}^T (see also [51,52]).

Theorem 15. Let $(M, g, \mathbf{x}^T, \lambda)$ be a Ricci soliton on a hypersurface of M of \mathbb{E}^{n+1} . Then, M is one of the following hypersurfaces of \mathbb{E}^{n+1} :

- (1) A hyperplane through the origin zero.
- (2) A hypersphere centered at the origin.
- (3) An open part of a flat hypersurface generated by lines through the origin zero;
- (4) An open part of a circular hypercylinder $S^1(r) \times \mathbb{E}^{n-1}$, $r > 0$;
- (5) An open part of a spherical hypercylinder $S^k(\sqrt{k-1}) \times \mathbb{E}^{n-k}$, $2 \leq k \leq n-1$,

where $n = \dim M$.

10. Interactions between Torqued Vector Fields and Ricci Solitons

In this section, we present some interactions between torqued vector fields and Ricci solitons on Riemannian manifolds from [40].

First, we recall the following definition.

Definition 4. The twisted product $B \times_f F$ of two Riemannian manifolds (B, g_B) and (F, g_F) is the product manifold $B \times F$ equipped with the metric:

$$g = g_B + f^2 g_F, \tag{45}$$

where f is a positive function on $B \times F$, which is called the twisting function. In particular, if the function f in (45) depends only B , then it is called a warped product, and the function f is called the warping function.

The following result from ([40], Theorem 2.1, p. 241) completely determined those Riemannian manifolds admitting torqued vector fields.

Theorem 16. *If a Riemannian manifold M admits a torqued vector field \mathcal{T} , then M is locally a twisted product $I \times_f F$ such that \mathcal{T} is always tangent to I , where I is an open interval. Conversely, for each twisted product $I \times_f F$, there exists a torqued vector field \mathcal{T} such that \mathcal{T} is always tangent to I .*

In view of Theorem 16, we made in [40] the following.

Definition 5. *A torqued vector field \mathcal{T} is said to be associated with a twisted product $I \times_f F$ if \mathcal{T} is always tangent to I .*

We have the following result from [40].

Theorem 17. *Every torqued vector field \mathcal{T} associated with a twisted product $I \times_f F$ is of the form:*

$$\mathcal{T} = \mu f \frac{\partial}{\partial s}, \tag{46}$$

where s is an arc-length parameter of I , μ is a nonzero function on F and f is the twisting function.

Theorem 18. *A torqued vector field \mathcal{T} on a Riemannian manifold M is a Killing vector field if and only if \mathcal{T} is a recurrent vector field that satisfies:*

$$\nabla_X \mathcal{T} = \alpha(X)\mathcal{T} \text{ and } \alpha(\mathcal{T}) = 0, \tag{47}$$

where α is a one-form.

As an application of Theorem 17, we have the following classification of torqued vector fields on Einstein manifolds.

Theorem 19. *Every torqued vector field \mathcal{T} on an Einstein manifold M is of the form:*

$$\mathcal{T} = \zeta Z, \tag{48}$$

where Z is a concircular vector field on M and ζ is a function satisfying $Z\zeta = 0$. Conversely, every vector field of the form (48) is a torqued vector field on M .

Another application of Theorem 17 is the following.

Corollary 8. *Up to constants, there exists at most one concircular vector field associated with a warped product $I \times_\eta F$.*

A Riemannian manifold (M, g) is called a quasi-Einstein manifold if its Ricci tensor Ric satisfies:

$$Ric = ag + b\alpha \otimes \alpha \tag{49}$$

for functions a, b , and one-form α .

A Riemannian manifold (M, g) is called a generalized quasi-Einstein [53] (resp., mixed quasi-Einstein [54] or nearly quasi-Einstein [55]) manifold if its Ricci tensor satisfies:

$$Ric = ag + b\alpha \otimes \alpha + c\beta \otimes \beta, \tag{50}$$

(resp., $Ric = ag + b\alpha \otimes \beta + c\beta \otimes \alpha$ or $Ric = ag + bE$)

where a, b, c are functions, α, β are one-forms and E is a non-vanishing symmetric $(0, 2)$ -tensor on M .
 In [40], we made the following definition.

Definition 6. A pseudo-Riemannian manifold is called almost quasi-Einstein if its Ricci tensor satisfies:

$$Ric = ag + b(\beta \otimes \gamma + \gamma \otimes \beta) \tag{51}$$

for some functions a, b and one-forms β and γ .

For Ricci solitons with torqued potential field, we have the following result from [40].

Theorem 20. If the potential field of a Ricci soliton $(M, g, \mathcal{T}, \lambda)$ is a torqued vector field \mathcal{T} , then (M, g) is an almost quasi-Einstein manifold.

The following result from [40] provides a very simple characterization for a Ricci soliton with torqued potential field to be trivial.

Theorem 21. A Ricci soliton $(M, g, \mathcal{T}, \lambda)$ with torqued potential field \mathcal{T} is trivial if and only if \mathcal{T} is a concircular vector field.

In view of Theorem 16, we made the following.

Definition 7. For a twisted product $I \times_f F$, the torqued vector field $f\partial/\partial s$ is called the canonical torqued vector field of $I \times_f F$, where s is an arc-length parameter on I .

We denote the canonical vector field $f\partial/\partial s$ by \mathcal{T}_{ca}^f .

Recall from Theorem 16 that if a Riemannian manifold M admits a torqued vector field, then it is locally a twisted product $I \times_f F$, where F is a Riemannian $(n - 1)$ -manifold and f is the twisting function. In [40], we proved the following.

Theorem 22. If $(I \times_f F, g, \mathcal{T}_{ca}^f, \lambda)$ is a Ricci soliton with the canonical torqued vector field \mathcal{T}_{ca}^f as its potential field, then we have:

- (a) \mathcal{T}_{ca}^f is a concircular vector field and
- (b) $(I \times_f F, g)$ is an Einstein manifold.

Remark 7. Ricci solitons (M, g, Z, λ) with concircular potential field Z have been completely determined in ([31], Theorem 5.1).

Remark 8. If the potential field of the Ricci soliton defined on $(I \times_f F, g)$ in Theorem 16 is an arbitrary torqued vector field \mathcal{T} associated with $I \times_f F$, then it follows from Theorem 17 that $\mathcal{T} = \mu f\partial/\partial s$ for some function μ defined on F . In this case, we may consider the twisted product $I \times_{\tilde{f}} \tilde{F}$ instead, where $\tilde{f} = \mu f$ and \tilde{F} is the manifold F with metric $\tilde{g}_F = \mu^{-2}g_F$. Then, $(I \times_{\tilde{f}} \tilde{F}, \tilde{g}, \mathcal{T}, \lambda)$ with $\tilde{g} = ds^2 + \tilde{f}^2\tilde{g}_F$ is a Ricci soliton whose potential field \mathcal{T} is the canonical torqued vector field $\mathcal{T}_{ca}^{\tilde{f}}$ of $I \times_{\tilde{f}} \tilde{F}$.

An important application of Theorem 22 is the following.

Corollary 9. Let $(I \times_f F, g, \mathcal{T}_{ca}^f, \lambda)$ be steady Ricci solitons with the canonical torqued vector field \mathcal{T}_{ca}^f as its potential field. If $\dim F \geq 2$, then we have:

- (a) \mathcal{T}_{ca}^f is a parallel vector field,
- (b) f is a constant, say c ,

- (c) $(I \times_c F, g)$ is a Ricci-flat manifold and
 (d) F is also Ricci-flat.

11. Conclusions

The position vector field x is the most elementary and natural object on a Euclidean submanifold. Similarly, the tangential component x^T of the position vector field is the most natural vector field tangent to the submanifold. From the results we mentioned above, we conclude that the tangential component x^T of the position vector field of the Euclidean submanifold is the most important vector field naturally associated with the Euclidean submanifold. The author believes that many further important properties of x^T can be proved.

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